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On Collocation Methods for Solving First-Order Volterra Type Linear Integro-Differential Equations

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Abstract

The numerical solutions of linear Volterra-type integro-differential equations (VIDEs) have been considered in this paper. We propose using the third kind of Chebyshev polynomial as the basis function to approximate the solution of the problems using **MAPLE 2018** software. Standard collocation points were chosen to collocate the approximate solution, and numerical experiments were performed on some sample problems already solved by the finite difference method and the method of power series as a basis polynomial, utilizing both the standard and Chebyshev-Gauss-Lobbatto collocation points. Furthermore, we compared our results to some previously published findings. Our proposed method yields superior approximate solutions and exhibits significantly lower absolute errors compared to the existing method. Furthermore, the absolute errors obtained are exceptionally minimal, indicating both convergence and computational efficiency.

Keywords: Chebyshev polynomial, Collocation, Approximate solution and Volterra Integro-differential equation

INTRODUCTION

This paper considers the collocation method for VIDEs, there has been extended research in recent years on integral and integro-differential equations for physical systems with memory effects in which stability and asymptotic stability have been the main interest. These numerical analyst aims to produce an efficient and effective method for obtaining a numerical solution to problems that prove difficult in getting their solution in a closed-form. There exist numerous numerical techniques to solve Integro-differential equation such as Wavelet-Galerkin Method (WGM) by (Avudainayan & Vani 2000), Homotopy Analysis Method (HAM) by (Kunjan Shah & Twinkle Singh, 2015).

Furthermore, the application of the Taylor, Chebyshev, Hermite, Legendre, and Laguerre polynomials and their numerical merits in solving integral and integro- differential equations (IDEs) numerically have been discussed in (Akyuz & Sezer, 2003), (Maleknejad & Mahmoudi, 2003), (Taiwo, O. A., Alim, A. T. & Akanmu, M. A., 2014), and (Richard & Roderick, 2010). Furthermore, many techniques such as a new algorithm for calculating Adomian polynomials, (Hashim, 2006), Chebyshev polynomials by (Eslahchi, M. R., Mehdi, D., Ayinde et al., 2021 & Sanaz, A., 2012), Chebyshev and Legendre by (Abubakar & Taiwo, 2014), Homotopy Perturbation Method (HPM), (Wazwaz, 2011) and, Variation Iteration Decomposition Method (VIDM) by (Ignatius & Mamadu, 2016). Application of Adomian's decomposition method on Integro- differential equation also examined by (Wazwaz, 2001) and others have been used to derive solutions of some classes of integro- differential equations. The great work did by the researchers aforementioned motivated us to develop a numerical approximation method that is efficient and accurate with less computational work to obtain an approximate solution for LVIDEs.

2. Basic Definitions

Definiion 2.1

An integro-differential equation is an equation in which the unknown function *g(x)* appears under the integral sign and contains an ordinary derivative $g^{(s)}$ as well. A standard integro-differential equation is of the form;

$$g^{(m)}(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,s)\, g(s)ds \qquad (1)$$

where $\alpha(x)$ and $\beta(x)$ are the limits of integration which may be constants, variables, or combined. λ is a constant parameter, $f(x)$ is a given function and $K(x, s)$ is a known function of two variables x and s called the kernel.

Definition 2.2

Volterra integro-differential equations (Wazwaz 2015)

Volterra integro-differential equations of the form:

$$g^{(m)}(x) = f(x) + \lambda \int_0^x K(x, s)\, g(s)ds \qquad (2)$$

where $g^{(m)}$ indicate the m^{th} derivative of $g(x)$, $K(x. s,)$, and function f(x) are given real-valued functions and λ is a constant parameter.

Definition 2.3

Chebyshev Polynomials:

The Chebyshev polynomial denoted by $T_n(x)$ and valid in the interval a ≤ x ≤ b is defined as

$$T_n(x) = \cos n \left[n \cos^{-1}\left(\frac{2x-(a+b)}{b-a}\right)\right], n = 0, 1, 2, \ldots \tag{3}$$

and the recurrence relation is given as

$$T_{n+1}(x) = 2\left[\frac{2x-(a+b)}{b-a}\right] T_n(x) - T_{n-1}(x), \quad a \le x \le b \tag{4}$$

Definition 2.4

Chebyshev polynomials of third kind (*Ayinde et al., 2022*)

The Chebyshev polynomial of the third kind in [-1, 1] of degree m is represented by $T_n(x)$ where:

$$T_n(x) = \cos\frac{\left(n+\frac{1}{2}\right)\vartheta}{\cos\left(\frac{\vartheta}{2}\right)}, \text{ where } x = \cos\vartheta \tag{5}$$

This elegance of Chebyshev polynomials satisfied the subsequent recurrence relation given by

$$T_0(x) = 1, \quad T_1(x) = 2x - 1, T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \qquad n = 2, 3, \cdots \tag{6}$$

The Chebyshev polynomial of the third kind in [a, b] of degree, m is represented by $T_n(x)$

$$T_n^*(x) = \cos\frac{\left(n+\frac{1}{2}\right)\vartheta}{\cos\left(\frac{\vartheta}{2}\right)}, \cos\vartheta = \frac{2x-(a+b)}{a-b}, \quad \vartheta \in [0, \pi] \tag{7}$$

Definition 2.5

Approximate solution: *(Ayinde et al. 2021)*

Approximate solution is an inexact representation of the exact solution that is still close enough to be used instead of exact and it is denoted by ζ(x), where is the degree of the approximant used in the calculation. Methods of the approximate solution are usually adopted because complete information needed to arrive at the exact solution may not be given. In this work, the approximate solution used is given as

$$\zeta(x) = \sum_{i=0}^{N} c_i \varphi_i(x) \tag{8}$$

where $Ci = 0, 1, 2, \ldots, M$ are unknown constants to be determined, $\varphi_i(x)$ $(i \ge 0)$ are the third kind of Chebyshev polynomials described in equations (5-7) and is the degree of approximating polynomials.

Definition 2.6

Exact Solution (*Ayinde et al., 2022*)

A solution is called an exact solution if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function, or the combination of two or more of these elementary functions.

3. Methodology

The general problem considered is of the form

$$a_{01}y^{n} + a_{11}y^{n-1} + a_{21}y^{n-2}+\ldots+a_{n1}y(x) + \lambda \int_{h(x)}^{i(x)} K(x, s)\, y(s)\, ds = f(x) \quad (9)$$

subject to boundary conditions.

$$y(b) = \text{A}, \; y(a) = \text{B} \quad (10)$$

where *h(x)* and *i(x)* are the limits of integration which may be constants, variables or combined, λ is a constants parameter, *f(x)* is a given function, and K (*x, s*) is called kernel.

In order to solve equations (9) and (10) using the standard method, we assumed an approximate solution of the form

$$y_n(x) = \sum_{i=0}^{N} a_i T_i^{*}(x) \quad i \geq 0 \quad (11)$$

where N is the degree of our approximant, *ai,* are the unknown constants to be determined and Ti, (*X*) are the third kind Chebyshev polynomials defined in equation above

Thus, differentiating equation (11) n times, we obtain

$$\left.\begin{aligned} y' &= \sum_{i=0}^{N} a_i T_i^{*\prime}(x) \\ y'' &= \sum_{i=0}^{N} a_i T_i^{*\prime\prime}(x) \\ &\;\vdots \\ y^{n} &= \sum_{i=0}^{N} a_i T_i^{*n}(x) \end{aligned}\right\} \quad (12)$$

Hence substituting (12) into equation (9) to obtain

$$a_{01}\sum_{i=0}^{N} a_i T_i^{*n}(x) + a_{11}\sum_{i=0}^{N} a_i T_i^{*(n-1)}(x) + \ldots + a_{n1}\sum_{i=0}^{N} a_i T_i^{*n}(x) + \lambda \int_{0}^{x} k(x,t)\left(\sum_{i=0}^{N} a_i T_i^{*n}(t)\right) dt = f(x) \quad (13)$$

Evaluating the integrals, we obtain

$$a_{01}\sum_{i=0}^{N} a_i T_i^{*n}(x) + a_{11}\sum_{i=0}^{N} a_i T_i^{*(n-1)}(x) + \ldots + a_{01}\sum_{i=0}^{N} a_i T_i^{*n}(x) + \lambda G(x) = f(x) \qquad (14)$$

where $G(x) = \int_0^x K(x,t)\left(\sum_{i=0}^{N} a_i T_i^{*n}(t)\right)dt$

Thus, collocation at point $X = Xk$, we obtain

$$a_{01}\sum_{i=0}^{N} a_i T_i^{*n}(x_k) + a_{11}\sum_{i=0}^{N} a_i T_i^{*(n-1)}(x_k) + \ldots + a_{n1}\sum_{i=0}^{N} a_i T_i^{*n}(x_k) + \lambda G(x_k) = f(x_k) \qquad (15)$$

where $x_k = a + \frac{(b-a)k}{N}$; k = 1, 2, . . ., N – 1.

Equation (15) gives rise to (N - 1) algebraic linear equation in (N + 1) unknown constants. Two extra equations are obtained from the conditions in equation (10). Altogether, we now have (N + 1) algebraic equations in (N + 1) unknown constants. These equations are then solved using Maple 2018 software to obtain the (N + 1) unknown constants $a_i(i \geq 0)$ which are then substituted back into the approximate solution given by equation (11).

4. Results

In this section, standard points have been employed to solve sample problems. The numerical solutions obtained using the present method had been compared with the exact solutions of the sample problems. Similarly, absolute errors of results from this present method have been compared with those obtained in Behrouz (2010) by finite difference method (FDM) and the use of power series as basis polynomial utilizing both the standard and Chebyshev-Gauss-Lobbatto collocation points for the same problems by Agbolade and Anake (2017). In the tables the following notations were used.

SCM_1: Solution via standard collocation using Power series as basis polynomial.
SCM_2: Solution via standard collocation using third kind Chebyshev polynomial as basis polynomial, which is the proposed method in this paper.

CGLCM: Solution via Chebyshev-Gauss-Lobatto collocation Method using third kind Power series basis polynomial.

Numerical Example 1

Consider the Linear Volterra linear integro-differential equation (Agbolade and Anake, 2017)

$$y'(x) = -y(x) - (x^2 - 2x + 1)e^{-x} + 5x^2 + 8 - \int_0^x t\, y(t)dt \qquad (12)$$

subject to initial condition

$$y(0) = 10 \quad (13)$$

The exact solution is given as $$y(x) = 10 - xe^x \quad (14)$$

Table 1.1 Comparison of exact solutions with approximate solutions for Numerical Example 1

x_i	**Exact**	**SCM$_2$**
0.0000	10.00000000	10.000000000
0.0714	9.933520218	9.933520218
0.1429	9.876128457	9.876128457
0.2143	9.827037138	9.827037139
0.2857	9.785299870	9.785299871
0.3571	9.750136229	9.750136229
0.4286	9.720801197	9.720801197
0.5000	9.696734670	9.696734670
0.5714	9.677310846	9.677310846
0.6429	9.661985366	9.661985366
0.7143	9.650325388	9.650325389
0.7857	9.641877524	9.641877522
0.8571	9.636254445	9.636254445
0.9286	9.633104130	9.633104128
1.0000	9.632120559	9.632120559

Table 1.2 Absolute Errors for Numerical Example 1

x_i	Error in SCM_1	Error in CGLCM (Agbolade & Anake (2017))	Error in FDM (Behrouz (2010))	Error in SCM_2
0.0000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.0714	$1.72431E-06$	$5.91262E-06$	$2.85397E-04$	0.0000000000
0.1429	$1.92637E-06$	$2.31105E-05$	$2.98284E-04$	0.0000000000
0.2143	$1.77825E-06$	$4.91013E-05$	$5.43393E-04$	1.0E-09
0.2857	$1.63695E-06$	$7.92123E-05$	$5.11413E-04$	1.0E-09
0.3571	$1.51288E-06$	$1.06967E-04$	$7.15638E-04$	0.0000000000
0.4286	$1.34028E-06$	$1.24865E-04$	$6.54200E-04$	0.0000000000
0.5000	$1.09939E-06$	$1.25617E-04$	$8.18261E-04$	0.0000000000
0.5714	$8.36590E-07$	$1.03879E-04$	$7.38321E-04$	0.0000000000
0.6429	$6.24770E-07$	$5.85256E-05$	$8.64022E-04$	0.0000000000
0.7143	$5.03018E-07$	$4.48751E-06$	$7.73248E-04$	1.0E-09
0.7857	$4.31615E-07$	$6.86526E-05$	$8.63249E-04$	2.0E-09
0.8571	$2.95402E-07$	$1.02783E-04$	$7.66939E-04$	0.0000000000
0.9286	$1.39661E-08$	$5.62088E-05$	$8.24573E-04$	2.0E-09
1.0000	$4.08829E-07$	$1.46741E-04$	$7.26353E-04$	0.0000000000

Numerical Example 2

Consider the Linear Volterra linear integro-differential equation (Agbolade and Anake, 2017)

$$y'(x) = -y(x) + \int_0^x e^{t-x}\, y(t)dt \qquad (15)$$

subject to initial condition

$$y(0) = 1 \quad (16)$$

The exact solution is given as $y(x) = e^{-x}coshx$ (17)

Table 1.3 Comparison of exact solutions with approximate solutions for Numerical Example 2

x_i	Exact	S SCM_2
0.0000	1.0000000000	0.9999862060
0.0833	0.9232690798	0.9232552092
0.1667	0.8582417719	0.8582278950
0.2500	0.8032653300	0.8032515254
0.3333	0.7567256737	0.7567120165
0.4167	0.7172846180	0.7172711789
0.5000	0.6839397204	0.6839265676
0.5833	0.6557119923	0.6556991912
0.6667	0.6317897828	0.6317773951
0.7500	0.6115650802	0.6115531624
0.8333	0.5944440975	0.5944326997
0.9167	0.5799345438	0.5799237081
1.0000	0.5676676417	0.5676573973

Table 1.4. Absolute Errors for Numerical Example 2

x_i	**Error in SCM_1 (Agbolade & Anake, 2017)**	**Error in CGLCM**	**Error in FDM**	**Error in SCM_2**
0.0000	0.0000	0.0000	0.0000	$1.37940E-05$
0.0833	$9.76008E-06$	$4.61438E-05$	$1.77203E-02$	$1.38706E-05$
0.1667	$9.82124E-06$	$7.55931E-05$	$2.16887E-03$	$1.38769E-05$
0.2500	$8.43856E-06$	$7.54254E-05$	$1.89273E-03$	$1.38046E-05$
0.3333	$7.74622E-06$	$4.08918E-05$	$4.52374E-03$	$1.36572E-05$
0.4167	$7.56304E-06$	$2.06182E-05$	$2.06181E-02$	$1.34391E-05$
0.5000	$7.32270E-06$	$8.83487E-05$	$7.13624E-03$	$1.31528E-05$
0.5833	$6.79994E-06$	$1.31400E-05$	$1.10585E-02$	$1.28011E-05$
0.6667	$6.15006E-06$	$1.18269E-05$	$8.20866E-03$	$1.23877E-05$
0.7500	$5.64809E-06$	$3.47095E-05$	$3.41335E-03$	$1.19178E-05$
0.8333	$5.40361E-06$	$8.79889E-05$	$8.16328E-03$	$1.13978E-05$
0.9167	$5.23390E-06$	$1.32084E-05$	$2.89396E-03$	$1.08357E-05$
1.0000	$4.79838E-06$	$1.62442E-05$	$3.27168E-03$	$1.02444E-05$

Discussion and Conclusion

Tables 1.1 and 1.3 display the numerical solutions obtained using the Volterra Integro-Differential Equations (VIDEs) solved with the basis function of third-kind Chebyshev polynomials. Comparing the absolute errors of the results obtained by the present method (SCM_2) with those obtained by the finite difference method for the same problems, it is evident that the method is both efficient and cost-effective for obtaining numerical solutions of first-order Volterra-type integro-differential equations. These findings are illustrated in Tables 1.2 and 1.4. Furthermore, the third-kind Chebyshev basis polynomial serves as an excellent approximation for these problems, producing results that compete favorably with existing methods.

Authors Contribution:

1. William D. (dunamakadiri@gmail.com)- Development of Scheme and Results
2. James A.A. - Implementation of schemes
3. Ayinde A. M. (ayinde.abdullahi@uniabija.edu.ng.) – Grammar & Plagiarism Checking
4. Oyedepo T. (oyedepotaiye@yahoo.com)- Editing & Typing of manuscript

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